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Exact solutions of the two-dimensional Burgers equation

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Abstract. Soliton-like solutions of the two-dimensional Burgers equation are obtained. Some new solutions from soliton-like solutions are also presented.

1. Introduction

Current interest in the area of the nonlinear evolution equation focuses on the need to find its soliton-like solutions, because the waveforms can change in different mechanisms and it usually has travelling wave solutions. In the present paper, we shall study the soliton-like solutions of the two-dimensional Burgers equation [1]

$$(u_t + uu_x - u_{xx})_x + u_{yy} = 0 \tag{1.1}$$

which describes weakly nonlinear two-dimensional shocks in dissipative media. The shocks described by equation (1.1) are weakly two-dimensional in the sense that the wave length of variation in the y direction is much larger than that in the x direction. Equation (1.1) can be referred to as the Zabolotskaya–Khoklov equation in nonlinear acoustics [2–5] and its Painlevé property and some exact solutions were given in [6].

In this paper we shall present the soliton-like solutions of equation (1.1). As special cases, we also obtain five types of exact solutions including a travelling wave solution and a steady-state solution etc.

2. Exact solutions

Recently, a direct method for finding soliton-like solutions [7–10] has been applied successfully to many nonlinear evolution equations. By using this direct method one can get not only the travelling wave solutions but also the non-travelling solitonic solutions.

We assume that equation (1.1) possesses solutions of the form

$$u(x, y, t) = A\partial_x^m \partial_y^n w[z(x, y, t)] + B \tag{2.1}$$

where the constants A, B and the integers m, n are to be determined. Using leading-order analysis [11], we can easily find that $m = 1$ and $n = 0$. Thus the solution can be chosen as

$$u(x, y, t) = A\partial_x w[z(x, y, t)] + B. \tag{2.2}$$

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By substituting (2.2) into (1.1) and with symbolic computation, we find that

$$\begin{aligned}
&w^{(3)}z_y^2 z_x + w'' z_{yy} z_x + w^{(3)} z_t z_x^2 + B w^{(3)} z_x^3 + A(w'')^2 z_x^4 + A w' w^{(3)} z_x^4 - w^{(4)} z_x^4 + 2w'' z_x z_{xt} \\
&\quad + 2w'' z_y z_{xy} + w' z_{xyy} + w'' z_t z_{xx} + 3Bw'' z_x z_{xx} + 5Aw' w'' z_x^2 z_{xx} - 6w^{(3)} z_x^2 z_{xx} \\
&\quad + A(w')^2 z_{xx}^2 - 3w'' z_{xx}^2 + w' z_{xxt} + Bw' z_{xxx} + A(w')^2 z_x z_{xxx} \\
&\quad - 4w'' z_x z_{xxx} - w' z_{xxxx} = 0.
\end{aligned} \tag{2.3}$$

Equating the coefficient of the highest power of z_x, i.e. z_x^4 terms, to zero, we obtain an ordinary differential equation

$$A(w'w'')' - w^{(4)} = 0 \tag{2.4}$$

which has a solution

$$w(z) = -\frac{2}{A}\ln(z). \tag{2.5}$$

Now we proceed to find the solution of equation (1.1) of the form given by (2.2) and (2.5) where $z(x, y, t)$ is expressed by following the x-linear form

$$z(x, y, t) = P(y, t) + \exp[Q(y, t)x + R(y, t)] \tag{2.6}$$

where $P(y, t)$, $Q(y, t)$ and $R(y, t)$ are differentiable functions with respect to y and t. After the substitution of equations (2.2), (2.5) and (2.6) with symbolic computation, we find that equation (1.1) becomes the equation

$$\begin{aligned}
&-(\mathrm{e}^{Qx+R})^2 Q_{yy} + \mathrm{e}^{Qx+R}[BPQ^3 - PQ^4 - Q^2P_t - 2PQQ_t + xPQ^2Q_t + PQ^2R_t + 2P_yQ_y \\
&\quad - 2xQP_yQ_y - 2xPQ_y^2 + x^2PQQ_y^2 - 2QP_yR_y - 2PQ_yR_y + 2xPQQ_yR_y \\
&\quad + PQR_y^2 + QP_{yy} - 2PQ_{yy} - xPQQ_{yy} - PQR_{yy}] \\
&\quad + [-BP^2Q^3 + P^2Q^4 + PQ^2P_t - 2P^2QQ_t - xP^2Q^2Q_t - P^2Q^2R_t \\
&\quad - 2QP_y^2 + 2PP_yQ_y + 2xPQP_yQ_y - 2xP^2Q_y^2 - x^2P^2QQ_y^2 + 2PQP_yR_y \\
&\quad - 2P^2Q_yR_y - 2xP^2QQ_yR_y - P^2QR_y^2 + PQP_{yy} - P^2Q_{yy} \\
&\quad - xP^2QQ_{yy} - P^2QR_{yy}] = 0.
\end{aligned} \tag{2.7}$$

Setting the coefficients of $(\mathrm{e}^{Qx+R})^2$, $\mathrm{e}^{Qx+R}x^2$, $\mathrm{e}^{Qx+R}x$, e^{Qx+R}, x^2, x and x^0 to zero, we obtain a set of constraints

$$\begin{cases}
Q_{yy} = Q_y = Q_t = 0 \\
BPQ^2 - PQ^3 - QP_t + PQR_t - 2P_yR_y + PR_y^2 + P_{yy} - PR_{yy} = 0 \\
BP^2Q^2 - P^2Q^3 - PQP_t + P^2QR_t + 2P_y^2 - 2PP_yR_y + P^2R_y^2 - PP_{yy} + P^2R_{yy} = 0.
\end{cases} \tag{2.8}$$

Hence under (2.8), the soliton-like solutions for equation (1.1) are obtained so that

$$\begin{aligned}
u(x, y, t) &= A\partial_x w[z(x, y, t)] + B = -\frac{2Q(y, t)\,\mathrm{e}^{Q(y,t)x+R(y,t)}}{P(y, t) + \mathrm{e}^{Q(y,t)x+R(y,t)}} + B \\
&= -Q(y, t)\left[1 + \tanh\frac{Q(y, t)x + R(y, t) - \ln P(y, t)}{2}\right] + B.
\end{aligned} \tag{2.9}$$

It follows from (2.8) that $Q(y, t) = k$, $k = \text{constant} \neq 0$, and now (2.8) becomes

$$\begin{cases}
k^2BP - k^3P - kP_t + kPR_t - 2P_yR_y + PR_y^2 + P_{yy} - PR_{yy} = 0 \\
k^2BP^2 - k^3P^2 - kPP_t + kP^2R_t + 2P_y^2 - 2PP_yR_y + P^2R_y^2 - PP_{yy} + P^2R_{yy} = 0.
\end{cases} \tag{2.10}$$

Under conditions (2.10), the formalism of the new soliton-like solutions of equation (1.1) becomes

$$u(x, y, t) = -k\left[1 + \tanh\frac{kx + R(y, t) - \ln P(y, t)}{2}\right] + B. \tag{2.11}$$

Thus we are able to construct new solutions of equation (1.1) by substituting the solutions of (2.10) into (2.11). As examples, we now consider the following three cases.

Case 1. When $P(y,t) = 1$ it is easy to deduce from (2.10) that

$$\begin{cases} R_{yy} + R_y^2 + kR_t + Bk^2 - k^3 = 0 \\ R_{yy} - R_y^2 - kR_t - Bk^2 + k^3 = 0. \end{cases} \tag{2.12}$$

In this case, we can obtain three types of solutions as follows:

(1) *Solitary waves.* Let us assume that $R(y,t) = ay + bt + c$, with arbitrary constants a, b and c. Substituting this into (2.10) we get $B = k - (b/k) - (a/k)^2$; therefore, the solution of (1.1) is of the form

$$u(x,y,t) = -\frac{b}{k} - \left(\frac{a}{k}\right)^2 - k \tanh \frac{kx + ay + bt + c}{2}. \tag{2.13}$$

Thus, solitary waves are nothing but a special case of the solution (2.11).

(2) *Solutions independent of y.* By setting $R(y,t) = a(t)$, we obtain from (2.12) that $a'(t) + Bk - k^2 = 0$, which has a solution $a(t) = k(k - B)t + c_1$. Thus, the corresponding solution of (1.1) is of the form

$$u(x,y,t) = -k\left[1 + \tanh \frac{kx + k(k-B)t + c_1}{2}\right] + B \tag{2.14}$$

where k, B and c_1 are arbitrary constants.

(3) *A general solution.* Choosing $R(y,t) = k^2 t + b(y)$, equation (2.12) reduces to the equation $b''(y) + (b'(y))^2 + Bk^2 = 0$, which has a general solution of the form $b(y) = \ln[\cos\sqrt{B}k(y - c_1)] + c_2$, with arbitrary constants c_1, c_2, k and B. So the solution of (1.1) is

$$u(x,y,t) = -k\left[1 + \tanh \frac{kx + k^2 t + \ln[\cos\sqrt{B}k(y - c_1)] + c_2}{2}\right] + B. \tag{2.15}$$

Case 2. When $P(y,t) = t$, equation (2.10) becomes

$$\begin{cases} (R_{yy} + R_y^2 + kR_t)t + (Bk^2 - k^3)t - k = 0 \\ (R_{yy} - R_y^2 - kR_t)t - (Bk^2 - k^3)t + k = 0. \end{cases} \tag{2.16}$$

Substituting $R(y,t) = \ln(y) + my$ with an arbitrary constant m into equation (2.16), we obtain $B = k - (m/k)^2$. This gives the *steady-state solution* of (1.1) as follows:

$$u(x,y,t) = -\left(\frac{m}{k}\right)^2 - k \tanh \frac{kx + my}{2}. \tag{2.17}$$

Case 3. When $P(y,t) = y$, equation (2.10) reduces to the equations

$$\begin{cases} y^2(R_{yy} + R_y^2 + kR_t) - 2yR_y + y^2(Bk^2 - k^3) + 2 = 0 \\ y(R_{yy} - R_y^2 - kR_t) + 2R_y - y(Bk^2 - k^3) = 0. \end{cases} \tag{2.18}$$

Choosing $R(y,t) = k(k - B)t + a(y)$, equation (2.18) is changed to the equation

$$y^2[a''(y) + (a'(y))^2] - 2ya'(y) + 2 = 0$$

which has solution $a(y) = \ln(y^2 - c_1 y) + c_2$, where c_1 and c_2 are arbitrary constants. So the solution of (1.1) is obtained as

$$u(x,y,t) = -k\left[1 + \tanh \frac{kx + k(k-B)t + \ln(y - c_1) + c_2}{2}\right] + B \tag{2.19}$$

where k and B are arbitrary constants.

If we set $R(y,t) = b(t)$ in case 2 and $R(y,t) = a(y)$ in case 3, then we can also obtain the travelling wave solution and the steady-state solution, respectively.

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